Research Reports of Oyama National
College of Technology, No.31(1999), 9−14

ON STRONGLY DIVISORIAL IDEALS OF SEMIGROUPS

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1. INTRODUCTION

Let S be a commutative additive semigroup with identity, that is, a *monoid.* The *iedntity* of a monoid S is denoted by symbol O. A monoid S is said to be *cancellative,* if $a+b=a+c$ with a,b,c in S means $b=c$ and S is said to be *torsion-free* if $ns=nt$ implies $s=t$ for all s,t in S and all positive integers n. In this paper we shall call a torsion-free cancellative monoid a *g-monoid.*

Let S be a g-monoid. Let $\sim$ be the equivalence relation on $S\times S$ defined by $(s_1, t_1)\sim(s_2, t_2)$ if $s_1+t_2=s_2+t_1$ and let $s-t$ be the equivalence class of (s, t) under $\sim$. Let $G=\{s-t \mid s,t\in S\}$ be the set of equivalence classes . Then G is evidently a commutative group and S is a submonoid of G.G is called the *quotient group* of S and is often denoted by $q(S)$. Each semigroup T lying between S and $q(S)$ is called an *oversemigroup* of S. Every oversemigroup T of a g-monoid S is evidently a g-monoid.

A non-empty subset I of a g-monoid S is called an *ideal* of S if $S+I\subseteq I$. An ideal I is properly contained in S , then I is called a *proper* ideal of S. A proper ideal P of S is called a *prime* ideal of S if $x+y\in P$ for x, y in S implies $x\in P$ or $y\in P$. An ideal I of S is said to be *finitely generated* if I can be expressible as $I=S+a_1\cup S+a_2\cup \dots \cup S+a_n$ for a finite number of elements $a_1, a_2, \dots, a_n$ of S.
In particular, if $I=S+a$ for some element $a\in S$, then I is called a *principal* ideal of S. In this paper, we shall denote the set of proper ideals of S by $I(S)$. An element $x\in S$ is called a *unit* of S if $x+y=0$ for some $y\in S$. Let $U(S)$ be the set of units of S. If $M=S\backslash U(S)$ is a non-empty subset of S, then M is the largest ideal of S and is called the *maximal ideal* of S.

Let S be a g-monoid with quotient group G. A nonempty subset I of G is called a *fractional ideal* of S if $S+I\subseteq I$ and $s+I\subseteq S$ for some $s\in S$. In this paper the set of fractional ideals of S is denoted by $F(S)$. For any $I\in F(S)$, we set $[S:I]=\{x\in G \mid x+I\subseteq S\}$. Then $[S:I]$ is also a fractional ideal of S , since $s+[S:I]\subseteq S$ for any $s\in I\cap S$. $[S:I]$ is called the *dual* of I. The dual of $I\in F(S)$ is also denoted by I^{-1} . A fractional ideal I of S is said to be *divisorial* if $I=I_v$ where $I_v=[S:I^{-1}]=(I^{-1})^{-1}$. Let $\boldsymbol{D}(S)$ denote the set of divisorial fractional ideals of S. For any $I\in F(S)$, $[I:I]=\{x\in G \mid x+I\subseteq I\}$ is an oversemigroup of S and is called a *conductor oversemigroup* of S.

In Section 2, we study strongly divisorial ideals of a g-monoid. In Theorem 7 and Proposition 9, we give two necessary and sufficient conditions of a fractional ideal to be strongly divisorial. In Section 3, we study the complete integral closure of a g-monoid. In Proposition 14, we give a representation of the complete integral closure of a g-monoid S with the aid of strongly divisorial ideals of S.

Throughout this paper, S will be a g-monoid and G will denote its quotient group. Unexplained terminology and unreferenced facts about semigroups may be found in [2].

2. STRONGLY DIVISORIAL IDEALS

A fractional ideals I of S is said to be *strong* if $I+I^{-1}=I$, that is, $I^{-1}=[I:I]$. If I is a strong fractional ideal, then $I\subseteq S$ and so I is an ideal of S. If I is an ideal of S, then $I\subseteq I+I^{-1}\subseteq S$ If $I+I^{-1}=S$ holds, then I is called an *invertible* ideal of S. If M is the maximal ideal of S, then M is either strong or invertible. Note that if $I^{-1}=S$, then $I+I^{-1}=S$ holds and so I is evidently invertible.

LEMMA 1. *Let I be a fractional ideal of S. Then I is a strong ideal if and only if $I=J+J^{-1}$ for some ideal J of S.*

PROOF. If I is strong, then $I=I+I^{-1}$. Conversely, assume that $I=J+J^{-1}$ for some ideal J of S. Then $I^{-1}=[S:(J+J^{-1})]=[[S:J]:J^{-1}]=[J^{-1}:J^{-1}]$. Hence $I+I^{-1}=(J+J^{-1})+[J^{-1}:J^{-1}]=J+(J^{-1}+[J^{-1}:J^{-1}])=J+J^{-1}=I$ as wanted.

REMARK 2. If F is a fractional ideal of S, then $F+F^{-1}$ is a strong ideal. In fact, since $s+F\subseteq S$ for some $s\in S$, if we set $J=s+F$, then J is an ideal of S and $F+F^{-1}=(s+F)+(-s+F^{-1})=(s+F)+(s+F)^{-1}=J+J^{-1}$. Hence $F+F^{-1}=J+J^{-1}$ is strong by Lemma 1.

REMARK 3. If I is an invertible ideal of S, then I^{-1} is not in general a subsemigroup of G. To see this, let $I=(a)$ with a nonunit a in S. Then clearly $-a\in I^{-1}$ but $-2a$ is not in I^{-1}. For, if $-2a\in I^{-1}$, then $-a=-2a+a\in I^{-1}+I=S$ and so a is a unit of S which contradicts the assumption.

LEMMA 4. *Let I be an ideal of S. Then the following statements are equivalent.*

(1) I^{-1} *is a subsemigroup of* G.

(2) $I^{-1}=[I_v:I_v]$.

(3) $I^{-1}=[(I+I^{-1}):(I+I^{-1})]$.

PROOF. (3)$\Rightarrow$(1). This is clear.

(1)$\Rightarrow$(2). Since I^{-1} is an oversemigroup of S, $I_v=[S:I^{-1}]$ is an ideal of I^{-1} and so we have $I^{-1}\subseteq[I_v:I_v]$. But $[I_v:I_v]\subseteq(I_v)^{-1}=I^{-1}$ also holds and hence $I^{-1}=[I_v:I_v]$.

(2)$\Rightarrow$(3). Since $(I+I^{-1})^{-1}=[(I+I^{-1}):(I+I^{-1})]$ by Lemma 1, it suffices to show that $I^{-1}=(I+I^{-1})^{-1}$. Clearly $(I+I^{-1})^{-1}\subseteq I^{-1}$. Conversely, let $u\in I^{-1}$ and $a+b\in I+I^{-1}$ where

$a \in I$ and $b \in I^{-1}$. Then $u+a+b = a+(u+b) \in I+I^{-1}$ and so $u \in [(I+I^{-1}):(I+I^{-1})]$. Hence $I^{-1} = (I+I^{-1})^{-1}$ as required.

PROPOSITION 5. *If I is an ideal of S, then I^{-1} is an oversemigroup of S if and only if I_v is strong.*

PROOF. This follows from Lemma 4, since $I^{-1} = (I_v)^{-1}$.

PROPOSITION 6. *If I is a strong ideal of S, then I_v is also a strong ideal.*

PROOF. Since I is strong, $I^{-1} = [I:I]$ is an oversemigroup of S. Then, by Lemma 4, $I^{-1} = [I_v : I_v]$ and therefore $(I_v)^{-1} = I^{-1} = [I_v : I_v]$, which implies that I_v is also strong.

As in [1], we call an ideal I of S *strongly divisorial* if I is strong and divisorial. If I_v is strong, then it is clear that I_v is strongly divisorial. Let $\boldsymbol{D}_s(S)$ denote the set of strongly divisorial ideals of S.

THEOREM 7. *Let I be a fractional ideal of S. Then I is strongly divisorial if and only if $I = (J+J^{-1})_v$ for some ideal J of S.*

PROOF. (If). Let J be an ideal of S. Then, by Lemma 1, $J+J^{-1}$ is strong and then $I = (J+J^{-1})_v$ is strongly divisorial by Proposition 6.

(Only if). Suppose that I is a strongly divisorial ideal of S. First, by Lemma 1, $I = J+J^{-1}$ for some ideal J of S and then , by hypothesis, $I = I_v = (J+J^{-1})_v$.

REMARK 8. (1) Let F be a fractional ideal of S and let s be an element of S such that $s+F \subseteq S$. If we set $J = s+F$, then , by Proposition 7, $(F+F^{-1})_v = (J+J^{-1})_v$ is strongly divisorial.

(2) Since $S^{-1} = S$, we have $S_v = (S^{-1})^{-1} = S$ and $S+S^{-1} = S+S = S$. Hence S is always strongly divisorial.

Let T be an oversemigroup of S. If we set $[S:T] = \{x \in G \mid x+T \subseteq S\}$, then $[S:T]$ is the maximum ideal of S that is still an ideal of T and ia called the *conductor* of S in T.

PROPOSITION 9. *Let I be a fractional ideal of S. Then I is strongly divisorial if and only if I is the conductor of S in some oversemigroup T of S.*

PROOF. If I is strongly divisorial, then I^{-1} is an oversemigroup of S and $[S:I^{-1}] = I_v = I$. Thus I is the conductor of S in I^{-1}. Conversely let $I = [S:T]$ for some oversemigroup T of S. Then $[I:I] = [S:T]:I = [S:(I+T)] = [S:I] = I^{-1}$ and so I is strong. Moreover,

$I = [S: T] = [S:[S:[S: T]]] = I_v$ and hence I is divisorial.

COROLLARY 10. *There is a one-to-one correspondence between the set X of oversemigrops of the form I^{-1} with integral ideals I of S and the set Y of strongly divisorial ideals J of S.*

PROOF. Let $\alpha: X \to Y$ be a map such that $\alpha(I^{-1}) = [S: I^{-1}] = I_v$ for each $I^{-1} \in X$. Then it follows from Proposotion 9 that α is well-defined and surjective, since if $I^{-1} \in X$, then $(I_v)^{-1} = I^{-1} = [I_v : I_v]$ and so I_v is strongly divisorial. Moreover, if $\alpha((I_1)^{-1}) = \alpha((I_2)^{-1})$ with $(I_1)^{-1}, (I_2)^{-1} \in X$, then $I_1^{-1} = ((I_1)_v)^{-1} = ((I_2)_v)^{-1} = I_2^{-1}$ and therefore α is injective.

PROPOSITION 11. *For each $I \in \boldsymbol{D}_s(S)$, we set $R = I^{-1}$. Then the map $\sigma: \boldsymbol{D}(R) \to \boldsymbol{D}(S)$ such that $\sigma(H) = (H+I)_v$ for any $H \in \boldsymbol{D}(R)$, is an injective map and we have $\sigma(\boldsymbol{D}_s(R)) \subseteq \boldsymbol{D}_s(S)$.*

PROOF. Clearly $H+I$ is an ideal of S for each $H \in \boldsymbol{D}(R)$, and so $(H+I)_v \in \boldsymbol{D}(S)$. Suppose now that $\sigma(H_1) = \sigma(H_2)$ with $H_1, H_2 \in \boldsymbol{D}(S)$. Then we have $\sigma(H_1):I = \sigma(H_2):I$. But $\sigma(H_1):I = (S:(S:(H_1+I))):I = (S:I):(S:(H_1+I)) = (S:I):((S:I):H_1) = R:(R:H_1) = H_1$, because $H \in \boldsymbol{D}(R)$. Likewise we have $\sigma(H_2):I = H_2$ and so we have $H_1 = \sigma(H_1):I = \sigma(H_2):I = H_2$ and hence σ is injective.

To prove $\sigma(H) \in \boldsymbol{D}_s(S)$ for each $H \in \boldsymbol{D}_s(S)$, it suffices to show that $(I+H)+(I+H)^{-1} = I+H$. Now it follows that $(I+H)+(I+H)^{-1} = (I+H)+S:(I+H) = (I+H) + ((S:I):H) = I+H+(R:H) = I+(H+(R:H)) = I+H$, because H is strongly divisorial in R.

EXAMPLE 1 2. Put $S = \{0, 2, 3, 4, 5, \cdots\} = \boldsymbol{Z}_0 - \{1\}$, where $\boldsymbol{Z}_0$ is the set of non-negative integers. Then S is a g-monoid and the quotient group of S is equal to $\boldsymbol{Z}$, the set of integters. If we set $I = \{4, 6, 7, 8, \cdots\}$, then I is an ideal of S. It is easy to see that $I^{-1} = \{0, 2, 3, 4, \cdots\} = S$ and then $I_v = S$. Hence I is not divisorial. Next we set $J = \{3, 4, 5, 6, 7, \cdots\}$. Then I is an ideal of S and $J^{-1} = \{-1, 0, 1, 2, 3, \cdots\}$. Then $J_v = \{3, 4, 5, 6, \cdots\}$ and so $J = J_v$ is divisorial. But $J + J^{-1} = \{2, 3, 4, 5, \cdots\} \neq J$ and therefore J is not strong.

3. THE COMPLETE INTEGRAL CLOSURE

An element x of G is said to be *almost integral* over S if $a+nx \in S$ for some $a \in S$ and all positive integers n. The set of elements of G that are almost integral over S is an oversemigroup of S and is called the *complete integral closure* of S. Here we denote the complete integral closure of S by S^*. S is said to be *completely integrally closed* if $S = S^*$.

PROPOSITION 13. *For any g-monoid* S *, we have* $S^* = \cup\{F:F \mid F \in F(S)\}$.

PROOF. $(\supseteq)$. Let $x \in F:F$ for any $F \in F(S)$. By definition, $s+F \subseteq S$ for some $s \in S$. If we set $J = s+F$, then J is an ideal of S. Then for any $d \in J$, we have $x+d \in x+J \subseteq J \subset S$ and then $2x+d = x+(x+d) \in x+J \subseteq J \subset S$. Continuation of this process leads us to obtain that $nx+d \in S$ for all positive integers n and hence $x \in S^*$.

$(\subseteq)$. If $x \in S^*$, then $d+nx \in S$ for some $d \in S$ and for all $n \geqq 1$. Set $J = \cup\{(d)+nx \mid n \geqq 1\}$. Then J is an ideal of S and $x+J \subseteq J$. Hence $x \in J:J \subseteq S^*$. This completes the proof.

PROPOSITION 14. *For each g-monoid S, we have* $S^* = \cup\{I^{-1} \mid I \in \boldsymbol{D}_s(S)\}$.

PROOF. For each $I \in \boldsymbol{D}_s(S)$, we have $I^{-1} = I:I$ and so the inclusion $\supseteq$ is clear.

To prove the reverse inclusion, let $x \in S^*$. Then $x \in F:F$ for some $F \in F(S)$ by Proposition 13. By Proposition 9, we have $I = S:[F:F] \in \boldsymbol{D}_s(S)$. Then $x \in [F:F] \subseteq [S:[S:[F:F]]] = I^{-1}$ with $I \in \boldsymbol{D}_s(S)$ and so our proof is complete.

COROLLARY 15. S *is completely integrally closed if and only if* S *is the unique strongly divisorial ideal of* S.

PROOF. By Proposition 14, $S = S^*$ if and only if $I^{-1} = S$ for each $I \in \boldsymbol{D}_s(S)$. But if $I^{-1} = S$, then $I = I_v = (I^{-1})^{-1} = S^{-1} = S$ and therefore $S = S^*$ if and only if $\boldsymbol{D}_s(S) = \{S\}$.

LEMMA 16. *Let* I *and* J *be two strongly divisorial ideals of* S. *Then there exists a strongly divisorial ideal* H *of* S *such that* $H \subseteq I \cap J$.

PROOF. Take $H = ((I+J)+(I+J)^{-1})_v$. Then H is a strongly divisorial ideal of S by Theorem 7. Now we have $[S:I] = [I:I] \subseteq [(I+J):(I+J)] \subseteq [[S:(I+J)]:[S:(I+J)]] = [S:((I+J)+(I+J)^{-1})] = [S:((I+J)+(I+J)^{-1})_v] = S:H$ and hence $I = I_v \supseteq H_v = H$. Similarly $[S:J] = [J:J] \subseteq [(I+J):(I+J)] \subseteq S:H$ and so $J = J_v \supseteq H_v = H$. Thus $H \subseteq I \cap J$ as wanted. This completes the proof.

If I and J are strongly divisorial ideals of S, then we denote by $(I \cdot J)$ the strongly divisorial ideal $H = [(I+J):(I+J)^{-1}]_v$ defined in Lemma 16.

PROPOSITION 17. *The following conditions are equivalent:*

(1) There exists a minimum strongly divisorial ideal of S.

(2) $S^* = I^{-1}$ *for some strongly divosorial ideal* I *of* S.

(3) $[S:S^*]$ *is not empty.*

PROOF. (1)⇒(2). Let I be a minimum element of $\boldsymbol{D}_s(S)$. Then we have $S^* = I^{-1}$ by Proposition 13.

(2)⇒(3). If $S^* = I^{-1}$ for some strongly divisorial ideal I of S, then $[S:S^*] = [S:I^{-1}] = I_v = I$ is not empty.

(3)⇒(1). Let $I = [S:S^*]$. Then, by Proposition 9, I is a strongly divisorial ideal of S. Now let J be an arbitrary strongly divisorial ideal of S. By Proposition 13, $J^{-1} \subseteq S^*$ and so $I = [S:S^*] \subseteq ((J)^{-1})^{-1} = J_v = J$. Thus I is a minimum element of $\boldsymbol{D}_s(S)$.

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(Received Sept. 30, 1998)