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ON PSEUDO-VALUATION MONOIDS

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1. INTRODUCTION.

An additive commutative semigroup with identity is called a *monoid*. A monoid S is said to be *cancellative* if $a + b = a + c$ with a, b, c in S, then $b = c$ and is said to be *torsion-free* if $na = nb$ with a, b in S and n an integer, then $a = b$. A monoid S is called a *g-monoid* if S is cancellative and torsion-free.

A non-empty subset I of a g-monoid S is called an *ideal* of S if $S + I \subseteq I$ holds. An ideal I of S is said to be *proper if* $I \neq S$. An ideal P of a g-monoid S is said to be *prime if* $a + b \in P$ with a and b in S, then $a \in P$ *or* $b \in P$. An element x of S is called a *unit* of S if $x + y = 0$ for some $y \in S$. Let $U(S)$ be the set of units of S. Then the set $M = S \setminus U(S)$ is a non-empty subset of S and is the largest ideal of S. The ideal M is called the *maximal ideal* of S. It is easily seen that the maximal ideal of a g-monoid S is a prime ideal. For every g-monoid S, the set $G = \{ s - t \in S \}$ is a commutative additive group and S is a submonoid of G. The group G is called the *quotient group* of S and is often denoted by $q(S)$. As in [3], a prime ideal P of a g-monoid S is said to be *strongly prime* if x, $y \in q(S)$ and $x + y \in P$ implies that $x \in P$ or $y \in P$.

In this paper we introduce the notion of a *pseudo-valuation monoid*. A g-monoid S is defined to be a *pseudo-valuation domain* if every prime ideal of S is strongly prime. Here we recall that A g-monoid S iscalled a *valuation monoid* if for every element x of $q(S)$,we have $x \in S$ or $-x \in S$.

It is shown that every valuation monoid is a pseudo-valuation monoid. The main purpose of this paper is to give some characterizations of a pseudo-valuation monoid.

2. PSEUDO-VALUATION MONOIDS.

We first show that every valuation monoid is a pseudo-valuation monoid.

Proposition 1. Every valuation monoid is a pseudo-valuation monoid.

Proof. Let V be a valuation monoid with quotient group G and let P be a prime ideal of V. Suppose that $x + y \in P$ with $x, y \in G$. If both x and y are in V, then we are done. Hence we may suppose that x is not in V. Then, by hypothesis, we have $-x \in V$ and therefore $y = (y + x) - x \in P$ as required. This completes the proof.

Lemma 2. Let P be a prime ideal of a g-monoid S with quotient group G. Then P is a strongly prime ideal of S if and only if $-x + P \subseteq P$ for each $x \in G \setminus S$, where $G \setminus S =$

$\{ x \in G \mid x$ is not in S $\}$.

Proof. First assume that P is strongly prime. If $x \in G \setminus S$ and $p \in P$, then $p = (p - x) + x \in P$. Hence $p - x \in P$ or $x \in P$. But since $x \in G \setminus S$, we have $p - x \in P$. *Thus* $-x + P \subseteq P$ as wanted. Conversely, assume that $-x + P \subseteq P$ for each $x \in G \setminus S$. Let $x + y \in P$ with $x , y \in G$. If x and y are in S, then clearly $x \in P$ *or* $y \in P$ and we are done .
Hence assume that x is not in S. Then by assumption, $-x + P \subseteq P$ *and so* $y = -x + (x + y) \in P$. This completes the oroof.

Proposition 3. The prime ideals of a pseudo-valuation monoid are linearly ordered.

Proof. Let P and Q be prime ideals of a pseudo-valuation monoid S with quotient gruop G and suppose $a \in P \setminus Q$. Then for each $b \in Q$, we have $a - b \in G \setminus S$. Hence by Lemma 2, we have $b - a + P \subseteq P$ and *so* $b = (b - a) + a \in P$. Hence we have $Q \subseteq P$.

Theorem 4. Let S be a g-monoid with maximal ideal M and quotient group G. Then the following statements are equivalent.

(1) S is a pseudo-valuation monoid.

(2) For any two ideals I and J of S, either $I \subseteq J$ or $J + M \subseteq I + M$.

(3) For any two ideals I and J of S, either $I \subseteq J$ or $J + M \subseteq I$.

(4) M is a strongly prime ideal.

Proof. (1)$\Rightarrow$(2). Assume that I is not contained in J and choose an element $a \in I \setminus J$. For each $b \in J$, $a - b \in G \setminus S$ and so by Lemma 2 , we have $b - a + M \subseteq M$. Thus $b + M \subseteq a + M \subseteq I + M$ and therefore $J + M \subseteq I + M$.

(2)$\Rightarrow$(3). This requires no comment

(3)$\Rightarrow$(4). By Lemma 2 , it suffices to show that if $a , b \in S$ with $a - b \in G \setminus S$, then $b - a + M \subseteq M$. Since $a - b \in G \setminus S$, (a) is not contained in $(b)$and hence by our hypothesis, $b + M \subseteq (a)$. Then $b - a + M \subseteq S$ and hence $b - a + M \subseteq M$ as wanted.

(4)$\Rightarrow$(1). Let $x \in G \setminus S$ and let P be a prime ideal of S. By Lemma 2, it suffices to show that $-x + P \subseteq P$. Since $P \subseteq M$, $-x + P \subseteq -x + M \subseteq M$. Then $-x + (-x + p) \in M$ for each $p \in P$. Then we have $2(-x + p) = (-2x + p) + p \in M + P \subseteq P$. But then, since $-x + p \in M \subseteq S$, $2(-x + p) \in P$ implies $-x + p \in P$. Thus $-x + P \subseteq P$ as wanted.

Let S be a g-monoid with quotient group G. Then each submonoid of G containing S is called an *overmonoid* of S.

Lemma 5. Let S be a pseudo-valuation monoid and let T be an overmonoid of S. If Q is a prime ideal of T , then every prime ideal of S contained in $Q \cap S$ is also a prime ideal of T.

Proof. Let P be a prime ideal of S such that $P \subseteq Q \cap S$. It suffices to show that $t + p \in P$ for all $t \in T$ and $p \in P$. Since $p = t + p - t \in P$, we have $t + p \in P$ or $-t \in P$, because P

is strongly prime. If $-t \in P \subseteq Q \cap S$, then $-t \in Q \subset T$ which implies that $-t$ is not a unit of T, a contradiction. Thus we must have $t + p \in P$ and so P is an ideal of T. Moreover, since P is strongly prime , P is a prime ideal of T. This completes the proof.

Theorem 6. Let S be a pseudo-valuation monoid. If T is an overmonoid of S such that $S \subset T$ satisfies incomparability, then T is also a pseudo-valuation monoid and Spec(T) $\subseteq$ Spec(S).

Proof. Let Q be a prime ideal of T. By Lemma 5, $Q \cap S$ is also a prime ideal of T. Then $Q \cap S$ and Q ly over $Q \cap S$ and so by incomparability, $Q = Q \cap S$. Hence Q is also a prime ideal of S. Moreover, Q is a strongly prime ideal of T because Q is a strongly prime ideal of S. Therefore T is a pseudo-valuation monoid and Spec (T) $\subseteq$ Spec (S).

Let S be a g-monoid and let I be an ideal of S. For each positive integer n, we set $nI = \{ a_1 + a_2 + \cdots + a_n \mid a_1, a_2, \cdots, a_n \in I \}$. Then nI ia also an ideal of S.

Proposition 7. Let S be a pseudo-valuation monoid and let I be an ideal of S. Then $P = \cap \{kI \mid k = 1, 2, \cdots \}$ is a prime ideal of S.

Proof. Let $x + y \in P$ with $x \in S \setminus P$ and $y \in S$. Then $x \in S \setminus nI$ for some integer $n > 0$. Hence, by Theorem 4, $2nI \subset (x)$ and then , for each positive integer k , we have $x + y \in P \subset (2n + k)I = 2nI + kI \subset (x) + kI$. Thus we have $y \in kI$ for all integers $k > 0$ and so $y \in P$ which implies that P is a prime ideal of S.

For each ideal I of a g-monoid S, we set $\mathrm{rad}(I) = \{ x \in S \mid nx \in I \text{ for some integer } n > 0 \}$. Then rad(I) is an ideal of S and is called the *radical* of I.

Corollary 8. Let I and J be ideals of a pseudo-valuation monoid S. If rad(J)is not contained in I, then J contains some power of I.

Proof. Suppose that J does not contain kI for each positive integer k. Then, by Theorem 4, we have $2J \subset kI$ for all integers $k > 0$ so that $2J \subseteq \cap \{ kI \mid k = 1, 2, \cdots \} = P$. Then P is a prime ideal of S by Proposition 7 and so $J \subseteq P \subseteq I$. Then rad($J$)$\subseteq P \subseteq I$, a contradiction.

Proposition 9. Let S be a pseudo-valuation monoid with maximal ideal M. If P is a nonmaximal prime ideal of S, then S_P is a valuation monoid, where $S_P = \{ x - s \mid x \in S \text{ and } s \in S \setminus P \}$.

Proof. Let G be the quotient group of S and let $x \in G$. *If* $x \in S$, *then* $x \in S_P$. Next assume $x \in G \setminus S$, then by Lemma 2, $-x + M \subset M$. Now choose $m \in M \setminus P$. Then $-x = -x + m - m \in M_P \subseteq S_P$. Hence S_P is a valuation monoid.

Theorem 10. Let S be a g-monoid with maximal ideal M and quotient group G. Then the following statements are equivalent.

(1) S is a pseudo-valuation monoid.

(2) S has the unique valuation overmonoid V with maximal ideal M.

(3) There exists a valuation overmonoid V such that every prime ideal of S is also a prime ideal of V.

Proof. (1)⇒(2). First note that there is a valuation overmonoid W with maximal ideal N such that $N \cap S = M$. By Lemma 5, M is also a prime ideal of W. Put $V = W_M$. Then clearly V is a valuation overmonoid with maximal ideal MW_M. By assumption, M is strongly prime and so we have $M = MW_M$.

(2)⇒(3). Let P be a prime ideal of S. Choose $p \in P$ *and* $v \in V$. Then, since $p \in M$, we have $v + p \in M$. Then $2v + p \in M$ and so $2(v + p) \in P$. Then we have $v + p \in P$ and hence P is an ideal of V. Now let $x + y \in P$ with $x, y \in V$. If x and y are in S, then clearly $x \in P$ *or* $y \in P$ and we are done. Hence assume that $x \in G \setminus S$. Then $x \in G \setminus M$ and so $-x \in V$. Since P is an ideal of V, we have $y = x + y - x \in P$. Therefore P is a prime ideal of V.

(3)⇒(1). Let V be the given valuation overmonoid. Let P be a prime ideal of S. Then P is a prime ideal of V and hence P is strongly prime. Hence S is a pseudo-valuation monoid.

Proposition 11. Let S be a pseudo-valuation monoid with maximal ideal M which is not a valuation monoid and let V be the valuation overmonoid of S in Theorem 10. If I is a principal ideal of S, then I is not an ideal of V.

Proof. Suppose that $I = a + S$ is an ideal of V. Then $I = I + V = a + S + V = a + V$. Choose $v \in V \setminus S$. Then $v + a \in I$ and so $v + a = s + a$ with $s \in S$ and hence $v = s \in S$ which is a contradiction.

Corollary 12. If a pseudo-valuation monoid S has a principal prime ideal, then S is a valuation monoid.

Proof. Assume that S is not a valuation monoid. Then there exists a valuation overmonoid $V \supset S$ with the same maximal ideal of S. If P is a principal prime ideal of S, then P is not an ideal of V by Proposition 11. But this contradicts Lemma 5. Hence S must be a valuation monoid.

REFERENCES

[1] R.Gilmer, Commutative Semigroup Rings, The Univ. of Chicago Press, 1984.

[2] R. Gilmer, Multiplicative Ideal Theory, Marcell Dekker, New York, 1972.

[3] J.R. Hedstrom and E.G. Houston, Pseudo-valuation domains, Pacific J. Math., 75 (1978) 137-147.

[4] A. Okabe , On strongly divisorial ideals of semigroups , Research Reports of Oyama National College of Technology , No. 31 (1999) , 9-14.

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