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# Admissible Bases of Transfer Matrices over UFDs and Their Applications

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**Abstract** In this paper, as a generalization of admissible GCD defined by Data and Hautus, the notion of admissible bases for transfer matrices of linear systems defined over unique factorization domains is introduced. Using this notion, a necessary and sufficient condition for the realizability of precompensators by static state feedbacks is presented.

**Key words** Transfer Matrices, Admissible Bases, Linear Systems over Rings, Unique Factorization Domains

## 1. Introduction

Linear systems defined over rings have been extensively studied in the last three decades (see e.g., [1][3][8][10][11][12] and the references therein). Linear systems over rings are a natural generalization of those over the real number field. For instance, linear systems over rings can be used for modeling systems characterized by parameters, systems described by time-delay differential equations, systems involving integration operators, and many others.

In this paper, we introduce a notion of admissible bases for transfer matrices of linear systems defined over unique factorization domains and, using this concept, study the problem of realizing precompensators by static state feedbacks.

The structure of this paper is as follows. Section 2 presents preliminaries, including some basic definitions, and important properties of commutative rings and of linear systems over rings. Section 3 gives the definition of admissible bases for matrices of linear systems, and presents a necessary and sufficient condition for the realizability of precompensators by static state feedbacks. Finally, in Section 4 some concluding remarks are given.

## 2. Preliminaries

In this section, basic definitions and important properties of commutative rings with identity will be summarized for the sake of easy readability. Further, linear systems defined over commutative rings will be briefly reviewed in terms of mathematical terminologies.

### 2.1 Mathematical Preliminaries

Throughout this study, $\boldsymbol{R}$ will denote the field of real numbers and $\mathcal{R}$ a commutative ring with identity 1. $\mathcal{R}$ is called a *domain* if for $a$, $b \in \mathcal{R}$, $a \neq 0$ and $b \neq 0$ imply that $ab \neq 0$, that is, there is no nonzero nilfac-

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tor in $\mathcal{R}$. An element $a \in \mathcal{R}$ is called a *unit* or an *invertible element* if there is a unique $b \in \mathcal{R}$ such that $ab = 1$. Then, the element $b$ is called the *inverse* of $a$ and is written as $b = a^{-1}$. A nonzero element $p \in \mathcal{R}$ is called *irreducible* if it is not a unit and if $p = ab$ for some $a,\ b \in \mathcal{R}$ implies that either $a$ or $b$ is a unit.

(2.1) **Definition**

A ring $\mathcal{R}$ is called a *unique factorization domain*(UFD) if it is a domain and if the following conditions are satisfied:

(i) Every $a(\neq 0) \in \mathcal{R}$ has a factorization of the form $a = p_1 \cdots p_r$ where $p_1, \cdots, p_r$ are irreducible elements in $\mathcal{R}$.

(ii) If $a = p_1 \cdots p_r = q_1 \cdots q_s$ are two such factorizations, then one has $r = s$ and by suitably permutating the indices $p_1 = \varepsilon_1 q_1,\ \cdots,\ p_r = \varepsilon_r q_r$ for some units $\varepsilon_i$ in $\mathcal{R}$. □

A well-known example of unique factorization domains is the ring $\mathcal{U}[x_1, \cdots, x_q]$ of all polynomials of indeterminates $x_1, \cdots, x_q$ with coefficients in a UFD $\mathcal{U}$. In particular, $\boldsymbol{R}[x_1, \cdots, x_q]$ is a UFD, and this fact is very suitable for studying linear systems over a UFD because systems characterized by parameters, systems described by time-delay differential equations and many others can be suitably described by systems over $\boldsymbol{R}[x_1, \cdots, x_q]$ with $q \geq 1$.

(2.2) **Definition**

A ring $\mathcal{R}$ is called a *principal ideal domain*(PID) if it is a domain and if for any ideal $\mathcal{A} \subseteq \mathcal{R}$ there exists an element $a \in \mathcal{R}$ such that $\mathcal{A}$ coincides with the ideal generated by $a$. □

It is well known that a field is a PID and a PID is a UFD. The ring $\boldsymbol{Z}$ of all integers is an example of PID's, and the ring $\boldsymbol{R}[x]$ of polynomials of a single indeterminate $x$ over $\boldsymbol{R}$ is also a PID, but the ring $\boldsymbol{R}[x_1, \cdots, x_q]$ with $q \geq 2$ is not.

(2.3) **Definition**

Let $\mathcal{R}$ be a commutative ring with identity 1 and $\mathcal{M}$ an additive abelian group. Then $\mathcal{M}$ is called an *$\mathcal{R}$-module* if a mapping $\mathcal{R} \times \mathcal{M} \ni (a,\ x) \longmapsto ax \in \mathcal{M}$, called *scalar multiplication*, is defined, which satisfies for all $a,\ b \in \mathcal{R}$ and $x,\ y \in \mathcal{M}$

(i) associative law: $(ab)x = a(bx)$,

(ii) distributive law: $(a+b)x = ax + bx$, $a(x+y) = ax + ay$, and

(iii) unitary law: $1x = x$. □

An element in an $\mathcal{R}$-module is often called a *vector*. A subset $\mathcal{N}$ of an $\mathcal{R}$-module $\mathcal{M}$ is said to be a *submodule* of $\mathcal{M}$ if for any $x,\ y \in \mathcal{N}$ and any $a \in \mathcal{R}$, $x + y \in \mathcal{N}$ and $ax \in \mathcal{N}$. A set $\{x_1, \cdots, x_k\}$ of nonzero elements of $\mathcal{M}$ is called *$\mathcal{R}$-linearly independent* or simply *linearly independent* if $\sum_{i=1}^{k} a_i x_i = 0,\ a_i \in \mathcal{R}$, implies $a_1 = \cdots = a_k = 0$.

(2.4) **Definition**

Let $\mathcal{M}$ be an $\mathcal{R}$-module. Then $\mathcal{M}$ is called a *free module* if there is a subset $\{u_1, \cdots, u_r\}$ of $\mathcal{M}$ such that $\{u_1, \cdots, u_r\}$ is linearly independent and generates the whole $\mathcal{M}$. In this case, $\{u_1, \cdots, u_r\}$ is called a *basis* of $\mathcal{M}$ and $r$ the *rank* of $\mathcal{M}$. □

(2.5) **Examples**

(i) $\mathcal{R}$ is a free $\mathcal{R}$-module of rank 1 with a basis $\{1\}$.

(ii) The set $\mathcal{R}^n := \{[a_1 \ \cdots \ a_n]^T \mid a_i \in \mathcal{R}\}$ of all $n$-tuple column vectors with entries in $\mathcal{R}$ is an $\mathcal{R}$-free module of rank $n$ with a basis $\{[1 \ 0 \ \cdots \ 0]^T, \cdots, [0 \ \cdots \ 0 \ 1]^T\}$ where $T$ denotes the transpose of matrices. □

(2.6) **Definition**

Let $\mathcal{M}$ and $\mathcal{M}'$ be $\mathcal{R}$-modules. A map $f : \mathcal{M} \longrightarrow \mathcal{M}'$ is called an *$\mathcal{R}$-homomorphism* if for any $x, \ y \in \mathcal{M}$ and any $a \in \mathcal{R}$

$$f(x+y) = f(x) + f(y), \quad f(ax) = af(x). \ \square$$

Let $\mathcal{M}$ and $\mathcal{M}'$ be free $\mathcal{R}$-modules of rank $n$ and $m$, respectively, and $\{u_i\}$, $\{v_j\}$ be their bases, respectively. Then any homomorphism $f : \mathcal{M} \longrightarrow \mathcal{M}'$ can be uniquely represented as an $m \times n$ matrix $A = (a_{ij})$ over $\mathcal{R}$, written $A \in \mathcal{R}^{m\times n}$, where $a_{ij}$ are uniquely determined by

$$f(u_i) = \sum_j a_{ij} v_j, \quad a_{ij} \in \mathcal{R}. \tag{2.7}$$

Conversely, any matrix $A = (a_{ij}) \in \mathcal{R}^{m\times n}$ over $\mathcal{R}$ defines a unique $\mathcal{R}$-homomorphism $f : \mathcal{M} \longrightarrow \mathcal{M}'$ through (2.7).

(2.8) **Remark**

It is well-known that if $\mathcal{R}$ is a UFD then for any $\{\xi_1, \cdots, \xi_q\} \subset \mathcal{R}$ there always exists a *greatest common divisor* (GCD) over $\mathcal{R}$ of $\xi_1, \cdots, \xi_q$. □

## 2.2 Linear Systems over Rings

Let $\mathcal{R}$ be a commutative ring with identity 1, $\mathcal{R}[z]$ denote the ring of polynomials of $z$ with coefficients in $\mathcal{R}$, and $\mathcal{R}(z)$ the ring of rational functions over $\mathcal{R}$. The set $\mathcal{R}(z)^m$ of all $m$-tuple column vectors with entries in $\mathcal{R}(z)$ is considered to be an $\mathcal{R}(z)$-linear space. A rational function $f(z) \in \mathcal{R}(z)$ is called *proper* or *causal* if it can be represented as $f(z) = p(z)/q(z)$ where $p(z), \ q(z) \in \mathcal{R}[z]$ such that $q(z)$ is a monic polynomial and deg $p(z) \le$deg $q(z)$. Let the set of all proper rational functions in $\mathcal{R}(z)$ be denoted by $\mathcal{P}(\mathcal{R})$ or simply by $\mathcal{P}$ if no confusion seems possible. A matrix $L(z) \in \mathcal{P}^{m\times m}$ is called *biproper* or *bicausal over $\mathcal{P}$* if $L(z)^{-1} \in \mathcal{P}^{m\times m}$.

Let $A \in \mathcal{R}^{\ n\times n}$, $B \in \mathcal{R}^{\ n\times m}$, $C \in \mathcal{R}^{\ r\times n}$ and $D \in \mathcal{R}^{\ r\times m}$. Then, by a system $\Sigma = (A, \ B, \ C, \ D)$ over $\mathcal{R}$, one means either one of the following systems:

(i) a continuous-time linear system of the form

$$\Sigma : \begin{cases} \dfrac{d}{dt}x(t) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t), \end{cases}$$

(ii) a discrete-time linear system of the form

$$\Sigma : \begin{cases} x(t+1) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) + Du(t), \end{cases}$$

where $u(t) \in \mathcal{R}^m$, $x(t) \in \mathcal{R}^n$ and $y(t) \in \mathcal{R}^r$ are the input, the state and the output of the system, respectively. Obviously, when a continuous-time linear system of the form (i) is considered, it is assumed that the time derivative $dx(t)/dt$ is defined in a suitable way.

The $r \times m$ matrix $H(z)$ given by

$$H(z) = C(zI - A)^{-1}B + D \tag{2.9}$$

is called the *transfer matrix* of $\Sigma$. Clearly, $H(z)$ is a matrix such that its all entries are proper rational functions of $z$, i.e., $H(z)$ is a matrix belonging to $\mathcal{P}^{r\times m}$.

Conversely, for any given matrix $H(z) \in \mathcal{P}^{r\times m}$, does there exist a system $\Sigma = (A, B, C, D)$ over $\mathcal{R}$, where $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, $C \in \mathcal{R}^{r\times n}$, $D \in \mathcal{R}^{r\times m}$, such that $H(z) = C(zI - A)^{-1}B + D$ ? Such a system $\Sigma = (A, B, C, D)$, if it exists, is called a *realization* over $\mathcal{R}$ of $H(z)$, and any matrix $H(z) \in \mathcal{P}^{r\times m}$ which has a realization is called a *transfer matrix* and is refered to simply as a *system* over $\mathcal{R}$. The rank $n$ of the state module $\mathcal{R}^n$ for a realization is called the *dimension* of the realization. A realization with a minimal dimension is called a *minimal realization.*

A system $\Sigma = (A,\ B,\ C,\ D)$ over $\mathcal{R}$ is called *reachable* if the $\mathcal{R}$-homomorphism $M_r : \mathcal{R}^{nm} \longrightarrow \mathcal{R}^n$ defined by $M_r = [B\ AB\ \cdots\ A^{n-1}B]$ is surjective, and is called *observable* if the $\mathcal{R}$-homomorphism $M_o : \mathcal{R}^n \longrightarrow \mathcal{R}^{rn}$ defined by $M_o = [C^T\ A^TC^T\ \cdots\ {A^{n-1}}^T C^T]^T$ is injective. Then, a realization $\Sigma = (A, B, C, D)$ of a transfer matrix $H(z) \in \mathcal{P}^{r\times m}$ is called *canonical* if $\Sigma$ is reachable and observable.

It is well-known that if $\mathcal{R}$ is a field then any $H(z) \in \mathcal{P}^{r\times m}$ has a minimal realization $\Sigma = (A,\ B,\ C,\ D)$, and further that a realization is minimal if and only if it is canonical. However, for realizations over a general ring $\mathcal{R}$ this statement does not hold true. For instance, even if $\mathcal{R}$ is a PID a minimal realization is not necessarily canonical.

## 2.3 Examples of Linear Systems over Rings

(1) **Systems with Integer Coefficients.** A system of the form

$$\begin{cases} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t), \end{cases}$$

where $A,\ B,\ C,\ D$ are matrices over $\boldsymbol{Z}$, is a system over $\mathcal{R} = \boldsymbol{Z}$.

(2) **Parametrized Systems.** Let $\lambda := (\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_\alpha)$ be real parameters, where $\lambda_i \in \boldsymbol{R}$. Then a system of the form

$$\begin{cases} \dfrac{d}{dt}x(t) &= A(\lambda)x(t) + B(\lambda)u(t) \\ y(t) &= C(\lambda)x(t) + D(\lambda_1)u(t) \end{cases}$$

is a system over $\mathcal{R} = \boldsymbol{R}\,[\lambda]$.

(3) **Delay Systems.** Let $\sigma := (\sigma_1,\ \sigma_2, \cdots, \sigma_\beta)$ be delay operators, where $(\sigma_i f)(t) := f(t - \tau_i)$ $(\tau_i > 0)$. Then a system of the form

$$\begin{cases} \dfrac{d}{dt}x(t) &= A(\sigma)x(t) + B(\sigma)u(t) \\ y(t) &= C(\sigma)x(t) + D(\sigma)u(t) \end{cases}$$

is a system over $\mathcal{R} = \boldsymbol{R}[\sigma]$.

(4) **Systems with Integration Operators.** Let $\mu := (\mu_1,\ \mu_2, \cdots, \mu_\gamma)$ be integration operators, where $(\mu_i f)(t) := \int_{t-\tau_i}^{t} f(u)du$ $(\tau_i > 0)$. Then the system of the form

$$\begin{cases} \dfrac{d}{dt}x(t) &= A(\mu)x(t) + B(\mu)u(t) \\ y(t) &= C(\mu)x(t) + D(\mu)u(t) \end{cases}$$

is a system over $\mathcal{R} = \boldsymbol{R}\,[\mu]$.

(5) **More General Systems.** Define a set $\omega = (\omega_1,\ \cdots,\ \omega_\varepsilon)$ of operators by
$(\omega_i f)(t) =$ linear functional of $\{f(\tau) : \tau \leq t\}$

Then a system of the form

$$\begin{cases} \dfrac{d}{dt}x(t) &= A(\lambda,\ \omega)x(t)+B(\lambda,\ \omega)u(t) \\ y(t) &= C(\lambda,\ \omega)x(t)+D(\lambda,\ \omega)u(t) \end{cases}$$

is a system over $\mathcal{R} = \boldsymbol{R}[\lambda,\ \omega]$.

## 3. Admissible Bases of Transfer Matrices and State Feedback Realization of Precompensators

Throughout this section, the underlying commutative ring with identity is assumed to be a unique factorization domain(UFD), denoted by $\mathcal{U}$. The main reason this assumption is made is twofold. First, this assumption ensures that any matrix $H(z)$ over $\mathcal{P} = \mathcal{P}(\mathcal{U})$ has a realization $\Sigma = (A, B, C, D)$ over $\mathcal{U}$ [1][11] so that every proper rational matrix can be considered as a transfer matrix. Secondly, the class of systems over UFD's seems to be reasonably large enough to cover systems appearing in applications. For instance, linear systems polynomially dependent on parameters, linear systems described by time-delay differential equations, linear systems involving integration operators and many others including those characterized by their combinations can be described as linear systems over UFD's $\boldsymbol{R}[x_1, \cdots, x_q]$ with $q \geq 1$. In addition, there is a more technical reason that, as mentioned in Remark (2.8), for any set $\{\xi_1, \cdots, \xi_q\} \subset \mathcal{U}$ there always exists a GCD of $\xi_1, \cdots, \xi_q$ in $\mathcal{U}$. This property plays an important role in the factorization theory for transfer matrices of systems over UFD's.

For notational simplicity, the indeterminate $z$, such as in $H(z)$, will be often omitted when no confusion seems to be possible.

(3.1) **Definition**

A subset $\mathcal{D}$ of $\mathcal{U}[z]$ is said to be a *denominator set* if the following conditions are satisfied:

(i) $\mathcal{D}$ is *multiplicative*, i.e., $1 \in \mathcal{D}$ and if $p,\ q \in \mathcal{D}$ then $pq \in \mathcal{D}$.

(ii) Each polynomial $p \in \mathcal{D}$ is *monic*(therefore $0 \notin \mathcal{D}$).

(iii) $\mathcal{D}$ is *saturated*, i.e., if $p \in \mathcal{D}$ and $q$ is monic and divides $p$ then $q \in \mathcal{D}$.

(iv) There exists at least one element $a \in \mathcal{U}$ such that $z - a \in \mathcal{D}$. □

Clearly, the set of all monic polynomials with $\{1\}$ is a denominator set. The denominator set plays a very important role in examination of the stability of systems. In this section we denote the set of all proper rational functions having a representation of $p/q$, where $p$ and $q$ are polynomials and $q \in \mathcal{D}$ for a denominator set $\mathcal{D}$, by $\mathcal{P}$. It is well known that, once the ring $\mathcal{U}$ and the denominator set $\mathcal{D}$ have been chosen, $\mathcal{P}$ is a UFD[3].

(3.2) **Definition**

Let $H \in \mathcal{P}^{r\times m}$ be a transfer matrix, and $\mathcal{L}(H)$ denote the module generated by the columns of $H$. If $\mathcal{L}(H)$ is free with rank $m$, then a matrix $E = [e_1, \cdots, e_m]$, $e_i \in \mathcal{L}(H)$, is said to be a *admissible basis* of $H$ if $\{e_1, \cdots, e_m\}$ is a basis of $\mathcal{L}(H)$ and, there exist polynomial matrices $P \in \mathcal{U}[z]^{r\times r}$ and $Q \in \mathcal{U}[z]^{r\times r}$ such that

(i) $PE$ is a polynomial matrix.

(ii) there exists a polynomial matrix $K \in \mathcal{U}[z]^{k\times m}$ such that $QH = PEK$.

(iii) $P$ and $Q$ are coprime, i.e., $A = PB = QC$ implies $A = PQD$ for some $D$. □

The admissible basis of a transfer matrix defined above can be constructed as follows.

Let $q$ be a lest common denominator of the elements of $H$ and $Q := qI$, where $I$ denotes the identity matrix. Define $\bar{P} = QH$ and let $V = [v_1, \cdots, v_m] \in \mathcal{U}[z]^{r\times m}$, where $\{v_1, \cdots, v_m\}$ is a basis of $\mathcal{L}(QH)$. Then, there exists a polynomial matrix $K$ such that $\bar{P} = VK$. Chose $a \in \mathcal{U}$ such that $z - a \in \mathcal{D}$ and let $\mu$ denote the maximum degree of elements of $V$. Then,

$$E = \frac{1}{(z-a)^{\mu}} \cdot V$$

is an admissible basis of $H$, where $P$ can be taken as $P = (z-a)^{\mu} I$.

To apply this notion to the problem of realizing a precompensator by a state feedback, first, for system $\Sigma = (A, B, C, D)$ over a UFD $\mathcal{U}$, consider a *compensator* $(F(z),\ G(z))$ of the form

$$u = F(z)x + G(z)v, \tag{3.3}$$

where $F(z)$ and $G(z)$ are dynamical systems with dimensions such that the formula (3.3) is well defined, and $v$ is a new input. Then, it easily follows that the transfer matrix $H_{F,G}(z)$ of the resulting closed loop system $\Sigma_{F,G}$ is given by

$$H_{F,G}(z) = H(z)L_{F,G}(z) \in \mathcal{P}^{r\times m}, \tag{3.4}$$

where

(3.5)

$$L_{F,G}(z) := (I - F(z)H_S(z))^{-1}G(z) \in \mathcal{P}^{m\times m}$$

and the matrix $H_S(z)$, called the *input/state transfer matrix*, is defined to be

$$H_S(z) := (zI - A)^{-1}B \in \mathcal{P}^{n\times m} \tag{3.6}$$

(3.7) **Definition**

A compensator $(F(z),\ G(z))$ is called

(i) *regular*, if $G(z)$ is bicausal over $\mathcal{P}$;

(ii) a *precompensator*, if $F(z) = 0$;

(iii) *pure dynamic feedback*, if $G$ is *static*, i.e., $G$ is a constant matrix over $\mathcal{U}$;

(iv) *static state feedback*, if both $F$ and $G$ are static. □

The regularity defined in (i) above means that all possible output trajectories that can be produced by the original system can also be produced by the closed loop system.

The problem of realizing a precompensator by a regular state feedback form can be stated as follows: For a given transfer matrix $H(z) \in \mathcal{P}^{r\times m}$ and a given bicausal precompensator $L(z) \in \mathcal{P}^{m\times m}$ for $H(z)$, find, if it exists, a regular static state feedback(abbreviated by RSSF) $(F, G)$ such that $L(z) = L_{F,G}(z)$.

First, we quote the following theorem.

(3.8) **Theorem**[3][5]

Let $H \in \mathcal{P}^{r\times m}$ be any transfer matrix having a reachable realization $\Sigma = (A, B, C, D)$ with its dimension $n$, and $H_S := (zI - A)^{-1}B$ be the input/state transfer matrix. Then, a bicausal precompensator $L \in \mathcal{P}^{m\times m}$ for $H$ is realizable by an RSSF if and only if $u \in \mathcal{U}[z]^m$ and $H_S u \in \mathcal{U}[z]^n$ imply $L^{-1}u \in \mathcal{U}[z]^m$. □

Based on the above theorem, the following theorem will be shown.

**(3.9) Theorem**

Let $H \in \mathcal{P}^{r\times m}$ be an transfer matrix having a reachable realization $\Sigma = (A, B, C, D)$ with its dimension $n$, $H_S := (zI - A)^{-1}B$ be the input/state transfer matrix.

Then, a bicausal precompenstor $L \in \mathcal{P}^{m\times m}$ for $H$ is realizable by an RSSF if and only if $HL := E$ is an admissible basis of $H$.

Further, if $L$ is realizable, then an RSSF $(F, G)$ that realizes $L$ is given as

$$G = {N_0}^{-1},$$
$$F = {N_0}^{-1}[N_1\ N_2\ \cdots\ N_n][B\ AB\ \cdots\ A^{n-1}B]^{\dagger} \quad (1)$$

where $M^{\dagger}$ denotes a right-inverse of matrix $M$ and

$$L(z)^{-1} = N_0 + N_1z^{-1} + N_2z^{-2} + \cdots \quad (2)$$

**Proof**

To prove the first assertion, assume that $E = \{e_1, \cdots, e_k\} \in \mathcal{L}(H)$ is an admissible basis of $H$. In order to apply Theorem(3.8), we have to prove that $L^{-1}u$ is polynomial whenever $u$ and $H_S u$ are polynomial. Since $u$ and $Hu$ are polynomial implies $H_S u$ is polynomial, we show that the following stronger statement holds, that is, $L^{-1}u$ is polynomial vector whenever $u$ and $Hu$ are polynomial vectors.

From Definition(3.2), there exist coprime polynomial matrices $P$ and $Q$ such that $PE$ is polynomial and there is a polynomial matrix $K$ such that $QW = PEK$. Because $Hu$ can be written as $Hu = Q^{-1}QHu$ and $Q$ and $PE$ are coprime, hence for some polynomial vector $v, QWu = QPEv$. Since $E$ is a basis of $\mathcal{L}(H)$, we can take $E$ and $P$ as above. Let $p := (z-a)^{\mu}$, then we have that $L^{-1}u = E^{-1}Hu = p(QPE)^{-1}QWu$ is polynomial. This means that the sufficiency is proved. Since $E$ is a basis of $\mathcal{L}(H)$, by Theorem(3.8) the necessity is clear.

To show the second assertion, assume that $L$ is realizable by an RSSF $(F, G)$. Then, (3.5) and (2) lead to the relations

$$L(z)^{-1} = G^{-1}(I - FH_S(z))$$
$$= G^{-1} - G^{-1}FH_S(z) = N_0 + N_1z^{-1} + \cdots \quad (3)$$

Since $H_S(z)$ is strictly proper, it follows from (3) that

$$G = N_0^{-1},\ FH_S(z) = I - N_0^{-1}L(z)^{-1} \quad (4)$$

Now, noticing $H_S(z) = (zI - A)^{-1}B$ and expanding both sides of (4) in powers of $z^{-1}$ yield

$$F(Bz^{-1} + ABz^{-2} + \cdots)$$
$$= -N_0^{-1}(N_1z^{-1} + N_2z^{-2} + \cdots) \quad (5)$$

Since $\Sigma = (A, B, C, D)$ is a realization of $H(z)$ with dimension $n$, (5) is satisfied if and only if the first $n$ terms of both sides of (5) need to be equal. That is, the equality

$$F[B\ AB\ \cdots\ A^{n-1}B] = -N_0^{-1}[N_1\ N_2\ \cdots\ N_n] \quad (6)$$

is equivalent to (5). By the reachability of $\Sigma$, $[B\ AB\ \cdots\ A^{n-1}B] \in \mathcal{U}^{n\times nm}$ is a surjective homomorphism from $\mathcal{U}^{nm}$ to $\mathcal{U}^{n}$. So, for the standard basis $\{e_1, \cdots, e_n\}$ of $\mathcal{U}^n$, choose $n$ vectors $\xi_1, \ldots, \xi_n$ from $\mathcal{U}^{nm}$ such that $[B\ AB\ \cdots\ A^{n-1}B]\xi_i = e_i$. Then, the matrix $[\xi_i, \cdots, \xi_n] \in \mathcal{U}^{nm\times n}$ is a right inverse matrix of $[B\ AB\ \cdots\ A^{n-1}B]$ , and hence

$[B\ AB\ \cdots\ A^{n-1}B]$ has a right-inverse matrix, denoted by $[B\ AB\ \cdots\ A^{n-1}B]^{\dagger} \in \mathcal{U}^{nm\times n}$. Therefore, (6) gives
$F = -N_0^{-1}[N_1\ N_2\ \cdots\ N_n][B\ AB\ \cdots\ A^{n-1}B]^{\dagger} \in \mathcal{U}^{m\times n}$.
This completes the proof that $L$ is realizable by an RSSF $(F, G)$ given by (1). □

## 4. Concluding Remarks

The notion of admissible GCD given by Data and Hautus in [3] is defined as follows.

Let $w_1, \cdots, w_m \in \mathcal{P}$. A GCD $d$ of $w_1, \cdots, w_m$ is called *admissible* if there exist polynomials $q$ and $p$ such that $pd$ and $qw_i/pd$, $i = 1, \cdots, m$ are polynomials, and $p$ and $q$ are coprime in $\mathcal{U}[z]$.

Clearly, this is the case of Defination (3.2) when $r = 1$, $m = 1$. Hence, the notion of admissible basis is a generalization of the admissible GCD.

Various factorization approaches of transfer matrices for linear systems defined over the real number field have been studied and applied for various important control problems (see, e.g., [2][4][7][9][13] and the references therein). In particular, the factorization approach using stable transfer matrices has been thoroughly studied and has played an important role to develop a new control theory, called the $H_\infty$ control theory ([4][9][13]).

On the other hand, a general factorization theory for transfer matrices of linear systems over UFD was developed by H. Inaba, N. Ito and W. Wang in [6]. And as an application of this theory they had obtained a solution[6] to the problem of the realizability of precompensators stated as above. The method given in this paper is different from that in [6]. But, because $E$ is a basis of $\mathcal{L}(H)$, $H$ can be written as $H = EK$ for some $K$. This expression can be considered as another factorization of $H$ from that in [6]. And it seems that there is a possibility to further investigate various problems on systems over rings in this line. For instance, the stabilizability problem, various decoupling problems and some other design problems.

### References

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