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Analysis of Dynamics of a 2 DOF Non-linear Vehicle Model with First-order Averaging Method

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Abstract

In this report, the dynamics of a periodically forced two degrees of freedom vehicle model with non-linear spring and viscous damping is studied. With the first-order averaging method, the analytical expression for stability analysis of the system is derived. The results show that derived expression can be used to detect the chaotic motion.

I. Introduction

The quarter-car model which has two degrees of freedom is often used in studying the heave motion of the vehicle [1]-[2] or dynamic absorber [3]. Since many vehicle components such as suspensions and tires, have non-linear properties, dangerous instabilities or chaotic responses could be introduced. Usually, the chaotic responses of vehicle model are investigated with numerical simulation because it is difficult to obtain the analytic solution of the non-linear differential equations. In this report, we show that when the non-linear spring is described by the third-order polynomial function, the chaotic response of the quarter-car model can be predicted with the analytic expression.

II. The 2-DOF Vehicle Model with Nonlinear Spring and Viscous Damping

A. Motion equations

The quarter-car model examined is shown in Fig. 1. The unsprung and sprung masses M_u and M_s are connected by non-linear springs. The coordinates x_u and x_s represent the displacement of the masses M_u and M_s with respect to the ground. The nonlinear spring force f_{ks} and f_{kt} are expressed by a third-order polynomial function which is often used in description of the nonlinear spring property of a car suspension [5]. The system is excited by a harmonic displacement $x_r(t)$ and $x_r(t) = A\cos\omega t$, where A and ω are the forcing amplitude and frequency, respectively.

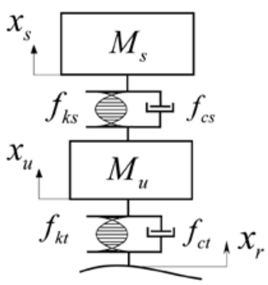

Fig. 1. 2-DOF Vehicle model

The governing equations of motion for the system are

$$M_s\ddot{x}_s = -f_{ks} - f_{cs} - M_s g \quad (1)$$
$$M_u\ddot{x}_u = f_{ks} + f_{cs} - f_{kt} - f_{ct} - M_u g \quad (2)$$

where

$$f_{ks} = k_{s1}(x_s - x_u - \delta_s) + k_{s2}(x_s - x_u - \delta_s)^2 + k_{s3}(x_s - x_u - \delta_s)^3 \quad (3)$$
$$f_{kt} = k_{t1}(x_u - A\sin(\omega t) - \delta_u) + k_{t2}(x_u - A\sin(\omega t) - \delta_u)^2 + k_{t3}(x_u - A\sin(\omega t) - \delta_u)^3 \quad (4)$$
$$f_{cs} = c_s(\dot{x}_s - \dot{x}_u) \quad (5)$$
$$f_{ct} = c_t(\dot{x}_u - A\omega\cos(\omega t)) \quad (6)$$

B. Dimensionless equations

Now we change the time scale from t to $\tau = \omega t$. Since $x = x(t)$ and $t = \tau/\omega$, we have

$$\ddot{x}_t = \omega^2 \ddot{x}_\tau \tag{7}$$

In the new time scale,

$$x_s(t) = x_s(\tau) = x_1, \quad \dot{x}_s(t) = \omega \dot{x}_s(\tau) = \omega \dot{x}_1, \quad \ddot{x}_s(t) = \omega^2 \ddot{x}_s(\tau) = \omega^2 \ddot{x}_1$$
$$x_u(t) = x_u(\tau) = x_2, \quad \dot{x}_u(t) = \omega \dot{x}_u(\tau) = \omega \dot{x}_2, \quad \ddot{x}_u(t) = \omega^2 \ddot{x}_u(\tau) = \omega^2 \ddot{x}_2$$

The motion equation of the system can be rewritten as

$$M_s\omega^2\ddot{x}_1 = -k_{s1}(x_1 - x_2 - \delta_s) - k_{s2}(x_1 - x_2 - \delta_s)^2 - k_{s3}(x_1 - x_2 - \delta_s)^3 - c_s(\omega\dot{x}_1 - \omega\dot{x}_2) - M_s g \tag{8}$$

$$\begin{aligned} M_u\omega^2\ddot{x}_2 = {} & k_{s1}(x_1 - x_2 - \delta_s) + k_{s2}(x_1 - x_2 - \delta_s)^2 + k_{s3}(x_1 - x_2 - \delta_s)^3 \\ & + c_s(\omega\dot{x}_1 - \omega\dot{x}_2) - k_{t1}(x_2 - A\sin\tau - \delta_u) - k_{t2}(x_2 - A\sin\tau - \delta_u)^2 - k_{t3}(x_2 - A\sin\tau - \delta_u)^3 \\ & - c_t(\omega\dot{x}_2 - A\omega\cos\tau) - M_u g \end{aligned} \tag{9}$$

Gathering the linear terms to the left side of the equations, we have

$$\ddot{x}_1 + \frac{c_s}{M_s\omega}\dot{x}_1 - \frac{c_s}{M_s\omega}\dot{x}_2 + \frac{k_{s1}}{M_s\omega^2}x_1 - \frac{k_{s1}}{M_s\omega^2}x_2 = -\frac{k_{s2}}{M_s\omega^2}(x_1 - x_2 - \delta_s)^2 - \frac{k_{s3}}{M_s\omega^2}(x_1 - x_2 - \delta_s)^3 + \frac{k_{s1}\delta_s - M_s g}{M_s\omega^2} \tag{10}$$

$$\begin{aligned} & \ddot{x}_2 - \frac{c_s}{M_u\omega}\dot{x}_1 + \frac{(c_s + c_t)}{M_u\omega}\dot{x}_2 - \frac{k_{s1}}{M_u\omega^2}x_1 + \frac{(k_{s1} + k_{t1})}{M_u\omega^2}x_2 = \frac{k_{s2}}{M_u\omega^2}(x_1 - x_2 - \delta_s)^2 + \frac{k_{s3}}{M_u\omega^2}(x_1 - x_2 - \delta_s)^3 \\ & + \frac{k_{t1}A}{M_u\omega^2}\sin\tau - \frac{k_{t2}}{M_u\omega^2}(x_2 - A\sin\tau - \delta_u)^2 - \frac{k_{t3}}{M_u\omega^2}(x_2 - A\sin\tau - \delta_u)^3 + \frac{c_t A}{M_u\omega}\cos\tau + \frac{k_{t1}\delta_u - (k_{s1}\delta_s + M_u g)}{M_u\omega^2} \end{aligned} \tag{11}$$

Let $y_i = x_i/A$ $(i = 1, 2)$, then

$$\begin{aligned} \ddot{y}_1 + \frac{c_s}{M_s\omega}\dot{y}_1 - \frac{c_s}{M_s\omega}\dot{y}_2 + \frac{k_{s1}}{M_s\omega^2}y_1 - \frac{k_{s1}}{M_s\omega^2}y_2 = & -\frac{Ak_{s2}}{M_s\omega^2}(y_1 - y_2 - \frac{\delta_s}{A})^2 - \frac{A^2k_{s3}}{M_s\omega^2}(y_1 - y_2 - \frac{\delta_s}{A})^3 \\ & + \frac{k_{s1}\delta_s - M_s g}{M_s\omega^2}\frac{1}{A} \end{aligned} \tag{12}$$

$$\begin{aligned} & \ddot{y}_2 - \frac{c_s}{M_u\omega}\dot{y}_1 + \frac{(c_s + c_t)}{M_u\omega}\dot{y}_2 - \frac{k_{s1}}{M_u\omega^2}y_1 + \frac{(k_{s1} + k_{t1})}{M_u\omega^2}y_2 = \frac{Ak_{s2}}{M_u\omega^2}(y_1 - y_2 - \frac{\delta_s}{A})^2 + \frac{A^2k_{s3}}{M_u\omega^2}(y_1 - y_2 - \frac{\delta_s}{A})^3 \\ & + \frac{k_{t1}}{M_u\omega^2}\sin\tau - \frac{Ak_{t2}}{M_u\omega^2}(y_2 - \sin\tau - \frac{\delta_u}{A})^2 - \frac{A^2k_{t3}}{M_u\omega^2}(y_2 - \sin\tau - \frac{\delta_u}{A})^3 + \frac{c_t}{M_u\omega}\cos\tau + \frac{k_{t1}\delta_u - (k_{s1}\delta_s + M_u g)}{M_u\omega^2}\frac{1}{A} \end{aligned} \tag{13}$$

To obtain the simplified form of equations (12) and (13), parameters related to $\dot{y}_k$ $(k = 1, 2)$ are introduced. The parameters related to $\dot{y}_k$ $(k = 1, 2)$ are defined as

$$\omega_{i,j} = \frac{\bar{\omega}_{i,j}}{\omega}, \qquad \bar{\omega}_{i,j}^2 = \frac{k_i}{M_j}, \qquad \xi_{i,j} = 2\zeta_{i,j}\omega_{i,j}, \qquad \zeta_{i,j} = \frac{c_i}{2\sqrt{k_i M_j}} \qquad (i = s, u; \quad j = s, u) \tag{14}$$

where $\omega_{i,j}$ are the dimensionless frequencies and $\bar{\omega}_{i,j}$ are the linear natural frequencies of the sprung and unsprung masses, respectively. Thus

$$\xi_{s,s} = 2\zeta_{s,s}\omega_{s,s} = 2\frac{c_s}{2\sqrt{k_s M_s}}\frac{\bar{\omega}_{s,s}}{\omega} = 2\frac{c_s}{2\sqrt{k_s M_s}}\frac{\sqrt{\frac{k_s}{M_s}}}{\omega} = \frac{c_s}{M_s\omega} \tag{15}$$

and

$$\xi_{s,u} = \frac{c_s}{M_u\omega}, \qquad \xi_{t,u} = 2\zeta_{t,u}\omega_{t,u} = \frac{c_t}{M_u\omega} \tag{16}$$

The parameters related to nonlinear terms are defined as:

$$\eta_{i,j,\ell} = \omega_{i,j}^2 A^\ell = \frac{\bar{\omega}_{i,j}^2}{\omega^2}A^\ell = \frac{1}{\omega^2}\frac{k_i}{M_j}A^\ell \quad (i = s1, s2, s3, t1, t2, t3; \quad j = s, u; \quad \ell = 0, 1, 2)$$

then

$$\frac{k_{s1}}{M_s\omega^2} = \frac{1}{\omega^2}\frac{k_{s1}}{M_s} = \frac{\bar{\omega}_{s1,s}^2}{\omega^2} = \omega_{s1,s}^2 A^0 = \eta_{s1,s,0}, \qquad \frac{k_{s1}}{M_u\omega^2} = \frac{1}{\omega^2}\frac{k_{s1}}{M_u} = \frac{\bar{\omega}_{s1,u}^2}{\omega^2} = \omega_{s1,u}^2 A^0 = \eta_{s1,u,0} \tag{17}$$

and

$$\frac{k_{t1}}{M_u\omega^2} = \eta_{t1,u,0}, \qquad \frac{Ak_{s2}}{M_u\omega^2} = \eta_{s2,u,1}, \qquad \frac{Ak_{s2}}{M_s\omega^2} = \eta_{s2,s,1}, \qquad \frac{A^2k_{s3}}{M_u\omega^2} = \eta_{s3,u,2}, \tag{18}$$

$$\frac{Ak_{t2}}{M_u\omega^2} = \eta_{t2,u,1}, \qquad \frac{A^2k_{t3}}{M_u\omega^2} = \eta_{t3,u,2}, \qquad \frac{A^2k_{s3}}{M_s\omega^2} = \eta_{s3,s,2} \tag{19}$$

The parameters related to constant terms are expressed as

$$\frac{\delta_s}{A} = \delta_1, \qquad \frac{k_{s1}\delta_s - M_s g}{M_s\omega^2}\frac{1}{A} = \delta_2, \qquad \frac{\delta_u}{A} = \delta_3, \qquad \frac{k_{t1}\delta_u - (k_{s1}\delta_s + M_u g)}{M_u\omega^2}\frac{1}{A} = \delta_4 \tag{20}$$

Taking expressions (15)-(20) into Eqs. (12)-(13) yields

$$\ddot{y}_1 + \xi_{s,s}\,\dot{y}_1 - \xi_{s,s}\,\dot{y}_2 + \eta_{s1,s,0}\,y_1 - \eta_{s1,s,0}\,y_2 = -\eta_{s2,s,1}\,(y_1 - y_2 - \delta_1)^2 - \eta_{s3,s,2}\,(y_1 - y_2 - \delta_1)^3 + \delta_2 \tag{21}$$

$$\begin{aligned}\ddot{y}_2 - \xi_{s,u}\,\dot{y}_1 + (\xi_{s,u} + \xi_{t,u})\,\dot{y}_2 - \eta_{s1,u,0}\,y_1 + (\eta_{s1,u,0} + \eta_{t1,u,0})\,y_2 = \eta_{s2,u,1}\,(y_1 - y_2 - \delta_1)^2 \\ +\eta_{s3,u,2}\,(y_1 - y_2 - \delta_1)^3 + \eta_{t1,u,0}\,\sin\tau - \eta_{t2,u,1}\,(y_2 - \sin\tau - \delta_3)^2 - \eta_{t3,u,2}\,(y_2 - \sin\tau - \delta_3)^3 \\ +\xi_{t,u}\,\cos\tau + \delta_4\end{aligned} \tag{22}$$

The equations (21) and (22) can be put in matrix form by taking nonlinear terms as the forcing functions and moving them to right side of the equations. They can be written as

$$M\ddot{\boldsymbol{y}} + C\dot{\boldsymbol{y}} + K\boldsymbol{y} = \boldsymbol{f} \tag{23}$$

where

$$\ddot{\boldsymbol{y}} = \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix}, \qquad \dot{\boldsymbol{y}} = \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix}, \qquad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

The mass, damping and stiffness matrices are expressed by

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} \xi_{s,s} & -\xi_{s,s} \\ -\xi_{s,u} & \xi_{s,u} + \xi_{t,u} \end{bmatrix}, \qquad K = \begin{bmatrix} \eta_{s1,s,0} & -\eta_{s1,s,0} \\ -\eta_{s1,u,0} & \eta_{s1,u,0} + \eta_{t1,u,0} \end{bmatrix}$$

respectively, while

$$\boldsymbol{f} = \begin{bmatrix} -\eta_{s2,s,1}\,(y_1 - y_2 - \delta_1)^2 - \eta_{s3,s,2}\,(y_1 - y_2 - \delta_1)^3 + \delta_2 \\ \\ \eta_{s2,u,1}\,(y_1 - y_2 - \delta_1)^2 + \eta_{s3,u,2}\,(y_1 - y_2 - \delta_1)^3 + \eta_{t1,u,0}\,\sin\tau \\ -\eta_{t2,u,1}\,(y_2 - \sin\tau - \delta_3)^2 - \eta_{t3,u,2}\,(y_2 - \sin\tau - \delta_3)^3 + \xi_{t,u}\,\cos\tau + \delta_4 \end{bmatrix}$$

C. Applying the method of averaging

The method of averaging (Krylov-Bogoliubov technique) is applied in seeking approximate steady state solutions of Eq. (23). The steady state responses are assumed as

$$\boldsymbol{y}(\tau) = \boldsymbol{u}(\tau)\cos\tau + \boldsymbol{v}(\tau)\sin\tau \tag{24}$$

where $\boldsymbol{u}(\tau) = [u_1(\tau)\;\; u_2(\tau)]^T$, $\boldsymbol{v}(\tau) = [v_1(\tau)\;\; v_2(\tau)]^T$ are taken as slow functions about the time τ. The motivation for this assumption is that when $\epsilon\boldsymbol{f}$ $(\epsilon \geq 0)$ is zero, equation (23) has its solutions of the form (24) and (25) with $\boldsymbol{u}(\tau)$ and $\boldsymbol{v}(\tau)$ constants. Then the velocity is expressed by

$$\dot{\boldsymbol{y}}(\tau) = -\boldsymbol{u}(\tau)\sin\tau + \boldsymbol{v}(\tau)\cos\tau \tag{25}$$

Now we seek a solution to Eq. (23) in the form of Eqs. (24) and (25). Differentiating Eq. (24) with respect to the time τ, we obtain

$$\dot{\boldsymbol{y}}(\tau) \;\; = \;\; \dot{\boldsymbol{u}}(\tau)\cos\tau - \boldsymbol{u}(\tau)\sin\tau + \dot{\boldsymbol{v}}(\tau)\sin\tau + \boldsymbol{v}(\tau)\cos\tau \tag{26}$$

Substituting Eq. (26) to (25) yields

$$\dot{\boldsymbol{u}}(\tau)\cos\tau + \dot{\boldsymbol{v}}(\tau)\sin\tau = 0 \tag{27}$$

Also differentiating Eq. (25)

$$\ddot{\boldsymbol{y}} = -\dot{\boldsymbol{u}}(\tau)\sin\tau - \boldsymbol{u}(\tau)\cos\tau + \dot{\boldsymbol{v}}(\tau)\cos\tau - \boldsymbol{v}(\tau)\sin\tau \tag{28}$$

Substituting expressions about $\ddot{\boldsymbol{y}}$, $\dot{\boldsymbol{y}}$ and $\boldsymbol{y}$, i.e. Eqs. (28), (25) and (24), into Eq. (23) and we have

$$(M\dot{\boldsymbol{v}} - M\boldsymbol{u} + C\boldsymbol{v} + K\boldsymbol{u})\cos\tau - (M\dot{\boldsymbol{u}} + M\boldsymbol{v} + C\boldsymbol{u} - K\boldsymbol{v})\sin\tau = f(\boldsymbol{u}, \boldsymbol{v}, \tau) \tag{29}$$

Equation (29) is multiplied by $-\sin\tau$

$$-(M\dot{\boldsymbol{v}} - M\boldsymbol{u} + C\boldsymbol{v} + K\boldsymbol{u})\cos\tau\sin\tau + (M\dot{\boldsymbol{u}} + M\boldsymbol{v} + C\boldsymbol{u} - K\boldsymbol{v})\sin^2\tau = f(\boldsymbol{u}, \boldsymbol{v}, \tau)(-\sin\tau) \tag{30}$$

Multiplying both sides of Eq. (27) by $M\cos\tau$

$$M\dot{\boldsymbol{u}}(\tau)\cos^2\tau + M\dot{\boldsymbol{v}}(\tau)\sin\tau\cos\tau = 0 \tag{31}$$

Adding Eqs. (31) and (30) and note $M\dot{\boldsymbol{u}}(\tau)\cos^2\tau + M\dot{\boldsymbol{u}}(\tau)\sin^2\tau = M\dot{\boldsymbol{u}}(\tau)$, we obtain

$$M\dot{\boldsymbol{u}}(\tau) = f(\boldsymbol{u},\boldsymbol{v},\tau)(-\sin\tau) - (M\boldsymbol{v}+C\boldsymbol{u}-K\boldsymbol{v})\sin^2\tau - (M\boldsymbol{u}-C\boldsymbol{v}-K\boldsymbol{u})\cos\tau\sin\tau \tag{32}$$

With the similar manipulation, we can obtain

$$M\dot{\boldsymbol{v}}(\tau) = f(\boldsymbol{u},\boldsymbol{v},\tau)\cos\tau + (M\boldsymbol{u}-C\boldsymbol{v}-K\boldsymbol{u})\cos^2\tau + (M\boldsymbol{v}+C\boldsymbol{u}-K\boldsymbol{v})\sin\tau\cos\tau \tag{33}$$

The right hand of Eqs. (32) and (33) will be integrated from 0 to 2π by assuming that $\boldsymbol{u}$ and $\boldsymbol{v}$ remain constant to obtain the approximated expression of $\dot{\boldsymbol{u}}$ and $\dot{\boldsymbol{v}}$.

D. Derivation of expression for $\boldsymbol{u}(\tau)$ and $\boldsymbol{v}(\tau)$

According to assumption expressed in Eq. (24),

$$\begin{aligned} y_1 &= u_1\cos\tau + v_1\sin\tau \\ y_2 &= u_2\cos\tau + v_2\sin\tau \end{aligned} \tag{34}$$

Thus nonlinear term $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y},\tau)$ in Eq. (23) can be expressed as

$$\boldsymbol{f} = \begin{bmatrix} -\eta_{s2,s,1}\left[(u_1-u_2)\cos\tau + (v_1-v_2)\sin\tau - \delta_1\right]^2 \\ -\eta_{s3,s,2}\left[(u_1-u_2)\cos\tau + (v_1-v_2)\sin\tau - \delta_1\right]^3 + \delta_2 \\ \\ \eta_{s2,u,1}\left[(u_1-u_2)\cos\tau + (v_1-v_2)\sin\tau - \delta_1\right]^2 \\ +\eta_{s3,u,2}\left[(u_1-u_2)\cos\tau + (v_1-v_2)\sin\tau - \delta_1\right]^3 + \delta_4 \\ +\eta_{t1,u,0}\ \sin\tau - \eta_{t2,u,1}\left(u_2\cos\tau + v_2\sin\tau - \sin\tau - \delta_3\right)^2 \\ +\xi_{t,u}\ \cos\tau + \xi_{t,u}\ \cos\tau - \eta_{t3,u,2}\left(u_2\cos\tau + v_2\sin\tau - \sin\tau - \delta_3\right)^3 \end{bmatrix} \tag{35}$$

Now we derive the expression for $\dot{\boldsymbol{u}}(\tau)$ which is given in Eq. (32). The right hand of Eq. (32) is integrated from 0 to 2π by assuing that $\boldsymbol{u}(\tau)$ and $\boldsymbol{v}(\tau)$ remain constant. Taking Eq. (35) into Eq. (32), then

$$\begin{aligned} M\dot{\boldsymbol{u}}(\tau) \ \approx\ & \frac{1}{2\pi}\int_0^{2\pi} \begin{bmatrix} -\eta_{s2,s,1}\left[(u_1-u_2)\cos\tau + (v_1-v_2)\sin\tau - \delta_1\right]^2 \\ -\eta_{s3,s,2}\left[(u_1-u_2)\cos\tau + (v_1-v_2)\sin\tau - \delta_1\right]^3 + \delta_2 \\ \\ \eta_{s2,u,1}\left[(u_1-u_2)\cos\tau + (v_1-v_2)\sin\tau - \delta_1\right]^2 \\ +\eta_{s3,u,2}\left[(u_1-u_2)\cos\tau + (v_1-v_2)\sin\tau - \delta_1\right]^3 + \delta_4 \\ +\eta_{t1,u,0}\ \sin\tau - \eta_{t2,u,1}\left(u_2\cos\tau + v_2\sin\tau - \sin\tau - \delta_3\right)^2 \\ +\xi_{t,u}\ \cos\tau + \xi_{t,u}\ \cos\tau - \eta_{t3,u,2}\left(u_2\cos\tau + v_2\sin\tau - \sin\tau - \delta_3\right)^3 \end{bmatrix} (-\sin\tau)\,d\tau \\ & -\frac{1}{2\pi}\int_0^{2\pi}(M\boldsymbol{v}+C\boldsymbol{u}-K\boldsymbol{v})\sin^2\tau\,d\tau - \frac{1}{2\pi}\int_0^{2\pi}(M\boldsymbol{u}-C\boldsymbol{v}-K\boldsymbol{u})\cos\tau\sin\tau\,d\tau \end{aligned} \tag{36}$$

Since $\int_0^{2\pi}\sin^2\tau\,d\tau = \pi$ and $\int_0^{2\pi}\cos\tau\sin\tau\,d\tau = 0$, integration of the second and the third terms of expression (36) become

$$-\frac{1}{2\pi}\int_0^{2\pi}(M\boldsymbol{v}+C\boldsymbol{u}-K\boldsymbol{v})\sin^2\tau\,d\tau \ = \ -\frac{1}{2}(M\boldsymbol{v}+C\boldsymbol{u}-K\boldsymbol{v}) \tag{37}$$

$$-\frac{1}{2\pi}\int_0^{2\pi}(M\boldsymbol{u}-C\boldsymbol{v}-K\boldsymbol{u})\cos\tau\sin\tau\,d\tau \ = \ 0 \tag{38}$$

and

$$M\dot{\boldsymbol{u}}(\tau) \approx -\frac{1}{2}(M\boldsymbol{v}+C\boldsymbol{u}-K\boldsymbol{v}) + \begin{bmatrix} -\frac{1}{8}(v_2-v_1)\left\{3c[(v_2-v_1)^2+(u_2-u_1)^2]+4b(3cb-2a)\right\} \\ \\ \frac{3}{8}f\,(v_2-v_1)\left[(v_2-v_1)^2+(u_2-u_1)^2\right] + \frac{3}{8}k\left(v_2^2+u_2^2\right)(v_2-1) \\ -\frac{6}{8}kv_2^2 + \left(\frac{9}{8}k + \frac{3}{2}kj^2 + \frac{3}{2}fb^2 - eb - ij\right)v_2 \\ +\left(eb - \frac{3}{2}fb^2\right)v_1 + \left(ij - \frac{3}{8}k - \frac{3}{2}kj^2 - \frac{1}{2}h\right) \end{bmatrix} \tag{39}$$

With the same procedure, we also have

$$M\dot{\boldsymbol{v}}(\tau) \approx \frac{1}{2}\left(M\boldsymbol{u} - C\boldsymbol{v} - K\boldsymbol{u}\right) + \begin{bmatrix} \frac{1}{8}(u_2-u_1)\left\{3c\left[(v_2-v_1)^2+(u_2-u_1)^2\right]-8ab+12cb^2\right\} \\ \frac{3}{8}f\left(u_1-u_2\right)^3+\frac{3}{8}f\left(v_1-v_2\right)^2\left(u_1-u_2\right)-\frac{3}{8}ku_2^3 \\ +\left(\frac{3}{4}kv_2-\frac{3}{8}kv_2^2+eb-\frac{3}{2}fb^2+ij-\frac{3}{2}kj^2-\frac{3}{8}k\right)u_2 \\ +\left(\frac{3}{2}fu_1b^2-eb\right)u_1+\frac{1}{2}L \end{bmatrix} \tag{40}$$

where $a=\eta_{s2,s,1}$, $b=\delta_1$, $c=\eta_{s3,s,2}$, $d=\delta_2$, $e=\eta_{s2,u,1}$, $f=\eta_{s3,u,2}$, $g=\delta_4$, $h=\eta_{t1,u,0}$, $i=\eta_{t2,u,1}$, $j=\delta_3$, $k=\eta_{t3,u,2}$ and $L=\xi_{t,u}$, respectively. From Eq. (23), we know

$$-\frac{1}{2}C\boldsymbol{u} = \begin{bmatrix} -\frac{1}{2}\xi_{s,s} & \frac{1}{2}\xi_{s,s} \\ \frac{1}{2}\xi_{s,u} & -\frac{1}{2}(\xi_{s,u}+\xi_{t,u}) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\xi_{s,s}u_1+\frac{1}{2}\xi_{s,s}u_2 \\ \frac{1}{2}\xi_{s,u}u_1-\frac{1}{2}(\xi_{s,u}+\xi_{t,u})u_2 \end{bmatrix} \tag{41}$$

Thus, Eq. (39) can be rewritten as

$$M\dot{\boldsymbol{u}}(\tau) = -\frac{1}{2}\left(M\boldsymbol{v} + C\boldsymbol{u} - K\boldsymbol{v}\right) = -\frac{1}{2}\left(M-K\right)\boldsymbol{v} + \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \tag{42}$$

where

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\xi_{s,s}u_1+\frac{1}{2}\xi_{s,s}u_2-\frac{1}{8}(v_2-v_1)\left\{3c[(v_2-v_1)^2+(u_2-u_1)^2]+4b(3cb-2a)\right\} \\ \frac{1}{2}\xi_{s,u}u_1-\frac{1}{2}(\xi_{s,u}+\xi_{t,u})u_2+\frac{3}{8}f\left(v_2-v_1\right)\left[(v_2-v_1)^2+(u_2-u_1)^2\right] \\ +\frac{3}{8}k\left(v_2^2+u_2^2\right)(v_2-1)-\frac{6}{8}kv_2^2+\left(\frac{9}{8}k+\frac{3}{2}kj^2+\frac{3}{2}fb^2-eb-ij\right)v_2 \\ +\left(eb-\frac{3}{2}fb^2\right)v_1+\left(ij-\frac{3}{8}k-\frac{3}{2}kj^2-\frac{1}{2}h\right) \end{bmatrix} \tag{43}$$

Note

$$-\frac{1}{2}C\boldsymbol{v} = -\frac{1}{2}\begin{bmatrix} \xi_{s,s} & -\xi_{s,s} \\ -\xi_{s,u} & \xi_{s,u}+\xi_{t,u} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\xi_{s,s}v_1+\frac{1}{2}\xi_{s,s}v_2 \\ \frac{1}{2}\xi_{s,u}v_1-\frac{1}{2}(\xi_{s,u}+\xi_{t,u})v_2 \end{bmatrix} \tag{44}$$

then Eq. (40) can be rewritten as

$$M\dot{\boldsymbol{v}} = \frac{1}{2}\left(M\boldsymbol{u} - C\boldsymbol{v} - K\boldsymbol{u}\right) = \frac{1}{2}\left(M-K\right)\boldsymbol{u} + \begin{bmatrix} Q_3 \\ Q_4 \end{bmatrix} \tag{45}$$

where,

$$\begin{bmatrix} Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\xi_{s,s}v_1+\frac{1}{2}\xi_{s,s}v_2+\frac{1}{8}(u_2-u_1)\left\{3c\left[(v_2-v_1)^2+(u_2-u_1)^2\right]-8ab+12cb^2\right\} \\ \frac{1}{2}\xi_{s,u}v_1-\frac{1}{2}(\xi_{s,u}+\xi_{t,u})v_2+\frac{3}{8}f\left(u_1-u_2\right)^3+\frac{3}{8}f\left(v_1-v_2\right)^2\left(u_1-u_2\right)-\frac{3}{8}ku_2^3 \\ +\left(\frac{3}{4}kv_2-\frac{3}{8}kv_2^2+eb-\frac{3}{2}fb^2+ij-\frac{3}{2}kj^2-\frac{3}{8}k\right)u_2+\left(\frac{3}{2}fu_1b^2-eb\right)u_1+\frac{1}{2}L \end{bmatrix} \tag{46}$$

III. Stability analysis

Equations (42) and (45) represent a system of four scalar, first order, autonomous, ordinary differential equations. Obviously, constant solution of the averaged system represented by Eqs. (42) and (45), correspond to 2π-periodic motions of the original system (23). The condition leading to these solutions is expressed by

$$\dot{\boldsymbol{u}} = \dot{\boldsymbol{v}} = 0 \tag{47}$$

Applying Eq. (47) in Eqs. (42) and (45) leads to a system of four coupled non-linear algebraic equations for u_i and v_i $(i=1,2)$.

$$\begin{aligned} -\frac{1}{2}\left(M-K\right)\boldsymbol{v}+\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} &= 0 \\ \frac{1}{2}\left(M+K\right)\boldsymbol{u}+\begin{bmatrix} Q_3 \\ Q_4 \end{bmatrix} &= 0 \end{aligned} \tag{48}$$

The stability analysis of a located periodic solutions of Eq. (23), say $\boldsymbol{u}_0=(u_{10}\ u_{20})^T$ and $\boldsymbol{v}_0=(v_{10}\ v_{20})^T$, is performed by letting

$$\boldsymbol{u}(\tau) = \boldsymbol{u}_0 + \boldsymbol{u}_1(\tau), \qquad \boldsymbol{v}(\tau) = \boldsymbol{v}_0 + \boldsymbol{v}_1(\tau) \tag{49}$$

TABLE I

Parameters of the numerical model

M_s	0.694	kg	M_u	0.353	kg
k_{s1}	517.8	N/m	k_{s2}	26028.0	N/m^2
k_{s3}	718349.0	N/m^3	k_{t1}	380.8	N/m
k_{t2}	19041.0	N/m^2	k_{t3}	563307.0	N/m^3
c_s	1.35	N·s/m	c_t	1.8	N·s/m
A	0.01	m			

where $\boldsymbol{u}_1(\tau) = (u_{11}\ u_{12})^T$ and $\boldsymbol{v}_1(\tau) = (v_{11}\ v_{12})^T$ are small perturbation from the periodic solution and $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$ are the steady state solution of Eq. (48).

$$\begin{aligned} u_1(\tau) &= u_{10} + u_{11}(\tau), \qquad & v_1(\tau) &= v_{10} + v_{11}(\tau) \\ u_2(\tau) &= u_{20} + u_{21}(\tau), \qquad & v_2(\tau) &= v_{20} + v_{21}(\tau) \end{aligned} \tag{50}$$

Substituting Eq. (50) into Eqs. (42) and (45), expanding the resulting equations in Taylor series with respect to u_{11}, u_{21}, v_{11}, and v_{21} and keeping terms up to the first order results in the linear system, the perturbed equations can be obtained which is a linear system in the from

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} \tag{51}$$

where

$$\boldsymbol{x} = [u_{11}(\tau)\ \ u_{21}(\tau)\ \ v_{11}(\tau)\ \ v_{21}(\tau)]^T \tag{52}$$

and

$$A = \begin{bmatrix} M & \mathbf{0} \\ \mathbf{0} & M \end{bmatrix}^{-1} \left(\begin{bmatrix} \mathbf{0} & -\frac{1}{2}(M-K) \\ \frac{1}{2}(M-K) & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \left.\frac{\partial Q_1}{\partial u_1}\right|_{\mathbf{0}} & \left.\frac{\partial Q_1}{\partial u_2}\right|_{\mathbf{0}} & \left.\frac{\partial Q_1}{\partial v_1}\right|_{\mathbf{0}} & \left.\frac{\partial Q_1}{\partial v_2}\right|_{\mathbf{0}} \\ \left.\frac{\partial Q_2}{\partial u_1}\right|_{\mathbf{0}} & \left.\frac{\partial Q_2}{\partial u_2}\right|_{\mathbf{0}} & \left.\frac{\partial Q_2}{\partial v_1}\right|_{\mathbf{0}} & \left.\frac{\partial Q_2}{\partial v_2}\right|_{\mathbf{0}} \\ \left.\frac{\partial Q_3}{\partial u_1}\right|_{\mathbf{0}} & \left.\frac{\partial Q_3}{\partial u_2}\right|_{\mathbf{0}} & \left.\frac{\partial Q_3}{\partial v_1}\right|_{\mathbf{0}} & \left.\frac{\partial Q_3}{\partial v_2}\right|_{\mathbf{0}} \\ \left.\frac{\partial Q_4}{\partial u_1}\right|_{\mathbf{0}} & \left.\frac{\partial Q_4}{\partial u_2}\right|_{\mathbf{0}} & \left.\frac{\partial Q_4}{\partial v_1}\right|_{\mathbf{0}} & \left.\frac{\partial Q_4}{\partial v_2}\right|_{\mathbf{0}} \end{bmatrix} \right) \tag{53}$$

where $\left.\frac{\partial Q_i}{\partial u_i}\right|_{\mathbf{0}}$, $\left.\frac{\partial Q_i}{\partial v_i}\right|_{\mathbf{0}}$, $(i = 1, 2, 3, 4;\ j = 1, 2)$ are first order Taylor expansion at $(\boldsymbol{u}_0, \boldsymbol{v}_0)$. Then by judge the eigenvalues of the coefficient determinant of Eq. (53), the stability o periodic solutions can be determined.

- If the real part of all the eigenvalues is negative, then the periodic solution is stable; otherwise, it is unstable.
- If a real eigenvalue changes sign, it is a saddle-node-type bifurcation and may result in jump phenomena.
- If there exists a pair of complex conjugate eigenvalues whose real part changes sign, it is termed Hopf bifurcation and results in quasiperiodic vibrations.

IV. Numerical example

In order to validate the derived equations for stability analysis, a case study was conducted. The parameters used are shown in Table I and they are from an experimental model [4]. Figure 2 show that responses of approximated system are close to ones obtained by direct integration of original system as $f = 2$ Hz. The results indicate that the expression (43) and (46) for $Q_i\ (i = 1, 2, 3, 4)$ are derived correctly.

The Poincaré maps of the responses of the system are shown in Fig.3. Each Poincaré map contains 10,000 sampling points and shows the existence of strange attractors. In this case, the computation of the corresponding eigenvalues of Eq. (53) gives $0.0041 \pm 0.4255i$ and $-0.0828 \pm 0.2222i$, which also indicates the existence of chaotic motions.

V. Concluding remarks

In this report, the analytical expression for stability analysis of a a two degrees of freedom non-linear vehicle model is given by applying the first order averaging method. The simplification was made in order to obtain Eq. (51). However, the analytical condition is still useful since it gives a guideline for parameter selection in dynamic design of the system. If more accurate expression is needed, the higher-order averaging method can be used and it is left to further studies.

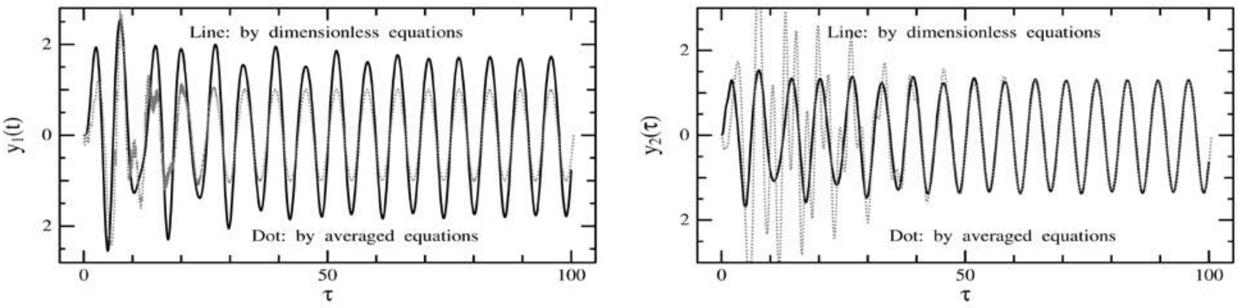

Fig. 2. System responses from dimensionless and averaging equations

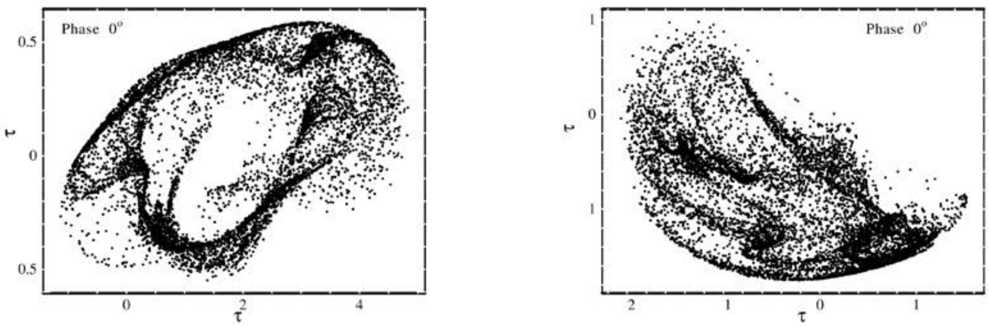

Fig. 3. Poincaré maps of chaotic motion of the system ($A = 0.01$ m, $f = 11.0$ Hz).

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